9/13/16

Differential Equations (§2.1–2.3)

$F(x, y, y', \cdots, y^{(n)}) = 0 \qquad n = \text{degree/order}$

Ex $\frac{dy}{dt} = ry$ $\qquad 1^{st}$ order $\qquad \frac{dy}{y} = r\,dt$

Solution $\int \frac{dy}{y} = \int r\,dt \qquad \ln|y| = rt + C_1$

$|y| = e^{rt} \cdot e^{C_1}$ or $y = \pm e^{C_1} e^{rt}$

General soln $\quad y = Ce^{rt}$

1. Separation of Variables

$$A(x) + B(y)\frac{dy}{dx} = 0$$

$$\Rightarrow dx\,A(x) = -B(y)\,dy \Rightarrow \int A(x)dx = -\int B(y)dy$$

...

Ex $\frac{dy}{dx} = \frac{xy}{1+x^2}$

Soln $\frac{dy}{y} = \frac{x}{1+x^2}dx$ $\qquad$ (variables separated!)

$$\int \frac{dy}{y} = \int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1}$$

$$\ln|y| = \frac{1}{2}\ln(x^2+1) + C_1 \Rightarrow y = \pm e^{C_1} e^{\sqrt{x^2+1}}$$

$$\therefore \boxed{y = Ce^{\sqrt{x^2+1}}}$$

Ex Solve the equation $\frac{dy}{dx} = \frac{\sin x}{y^2}$

Sol'n $y^2 dy = \sin x\, dx$

$$\int y^2 dy = \int \sin x\, dx \Rightarrow \frac{y^3}{3} = -\cos x + C_1$$

$$y^3 = -3\cos x + 3C_1 \Rightarrow \boxed{y = (-3\cos x + C)^{\frac{1}{3}}}$$

Ex ① Solve $y\frac{dy}{dx} = x^2$ ② $\frac{dy}{dx} = x^2 y$

Sol'n ① $y\, dy = x^2 dx$

$$\int y\, dy = \int x^2 dx \Rightarrow \frac{1}{2}y^2 = \frac{x^3}{3} + C_1$$

$$\therefore \quad y^2 = \frac{2}{3}x^3 + C \quad \text{or } y = \pm\sqrt{\frac{2}{3}x^3 + C}$$

② $\frac{dy}{y} = x^2 dx \qquad \int \frac{dy}{y} = \int x^2 dx$

$$\therefore \ln|y| = \frac{x^3}{3} + C_1 \Rightarrow \boxed{y = Ce^{x^3/3}}$$

Ex (Population) DE : $\frac{dx}{dt} = rx$ $\qquad r$ = growth rate

IC $x(0) = x_0$

Sol'n $\int \frac{dx}{x} = \int r\, dt \Rightarrow \ln|x| = rt + C_1$

$$\Rightarrow x(t) = ce^{rt} \Rightarrow \boxed{x(t) = x_0 e^{rt}}$$

Ex. An island's population follows the exp. growth model. The current population is 2000, and annual growth rate is 3%. How long does it take to double the pop.?

Soln

$$\begin{cases} \frac{dx}{dt} = 0.03x \\ x(0) = 2000 \end{cases} \Rightarrow x(t) = 2000e^{0.03t}$$

$$4000 = 2000e^{0.03t} \Rightarrow 2 = e^{0.03t}$$

$$\ln 2 = 0.03t \Rightarrow t = \frac{\ln 2}{0.03} \approx \frac{0.69}{0.03} = 23 \text{(yr)}$$

68-70-72 rule

Read Ex 3 (p 46)

Linear Differential Equations

(i) Constant coefficient case: $a\dot{x} + bx = 0$ (1^{st} ord)

$\dot{x} \equiv x' \equiv \frac{dx}{dt}$ $\qquad a\ddot{x} + b\dot{x} + cx = 0$ (2^{nd})

Ex Solve: $4x' + x = 0$ $\qquad 4\frac{dx}{dt} + x = 0$

Soln $\quad 4\lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{4} \quad x = e^{-\frac{1}{4}t}$

Gen. soln $\quad x = ce^{-\frac{1}{4}t}$

Ex $ax'' + bx' + cx = 0$

Sol'n Try $x = e^{\lambda t}$ $\quad a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c e^{\lambda t} = 0$

char. poly $\Rightarrow$ $\boxed{a\lambda^2 + b\lambda + c = 0}$

$\lambda_1, \lambda_2 \Rightarrow$ $\boxed{x = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}}$ ▨

Rem : Compare with $ax_{n+2} + bx_{n+1} + cx_n = 0$

$a\lambda^2 + b\lambda + c = 0 \Rightarrow x = \alpha\lambda_1^n + \beta\lambda_2^n$.

Three situations

(i) Two real char. roots λ_1, λ_2

$$x(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}$$

(ii) Double (real) root λ

$$x(t) = \alpha e^{\lambda t} + \beta t e^{\lambda t}$$

(iii) Two complex roots $\lambda_{1,2} = a \pm bi$

$$x(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

Ex. Solve $\begin{cases} \ddot{x}-2\dot{x}-3x=0 \\ x(0)=1,\ \dot{x}(0)=2 \end{cases}$

Sol'n Char equ: $\lambda^2-2\lambda-3=0$

$(\lambda-3)(\lambda+1)=0 \Rightarrow \lambda=3,-1$

$x=c_1e^{3t}+c_2e^{-t}$, $\dot{x}=3c_1e^{3t}-c_2e^{-t}$

IC: $\begin{cases} c_1+c_2=1 \\ 3c_1-c_2=2 \end{cases}$

$c_1=\dfrac{\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}}{\begin{vmatrix}1 & 1\\ 3 & -1\end{vmatrix}}=\dfrac{-3}{-4}=\dfrac{3}{4}$, $c_2=\dfrac{\begin{vmatrix}1 & 1\\ 3 & 2\end{vmatrix}}{-4}=\dfrac{1}{4}$.

$\therefore$ $\boxed{x=\frac{3}{4}e^{3t}+\frac{1}{4}e^{-t}}$

(ii) General linear 1^{st} order diff equ

$$\frac{dx}{dt}+p(t)x=q(t)$$

(1^{st} order gen. coeff.)

Idea: $m(t)\dfrac{dx}{dt}+m(t)p(t)x=m(t)q(t)$

Multiplier $e^{\int p(t)dt}$

$$e^{\int p(t)dt}\left(\frac{dx}{dt}+p(t)x\right)=e^{\int p(t)dt}q(t)$$

$$\parallel$$

$$\frac{d}{dt}\left(x e^{\int p(t)dt}\right)=e^{\int p(t)dt}q(t)$$

$$\Rightarrow \quad x e^{\int p(t)dt}=\int e^{\int p(t)dt}q(t)\ldots$$

<u>Ex</u> $\frac{dx}{dt}-2tx=t$

<u>Sol'n</u> multiplier $e^{\int -2t\,dt}=e^{-t^2}$

$$e^{-t^2}\left(\frac{dx}{dt}-2tx\right)=te^{-t^2}$$

$$\frac{d}{dt}\left(xe^{-t^2}\right)=te^{-t^2}$$

$$\Rightarrow \quad xe^{-t^2}=\int te^{-t^2}dt=\frac{1}{2}\int e^{-t^2}d(t^2)=-\frac{1}{2}e^{-t^2}+c_1$$

$$\Rightarrow \quad \boxed{x=-\frac{1}{2}+ce^{t^2}}$$